On the Monge and Beckmann Formulations of Optimal Transport in Abstract Wiener Spaces

Andrii Bobrov (Kyiv Academic University)

1 October 2026
Fields Institute

Monge Problem on $\mathbb{R}^n$

Let μ and ν be two probability measures defined on a Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$.
Recal that for a measurable function $T : \mathbb{R}^n \to \mathbb{R}^n$, the *push-forward* (or an *image measure*) $T_{\#}\mu$ of a measure μ under the image T is defined as

$$\forall A \in \mathcal{B}(\mathbb{R}^n) : T_{\#}\mu(A) = \mu\left(T^{-1}(A)\right)$$

Definition (Monge problem)

For two probability measures μ, ν find a measurable map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T_{\#}\mu = \nu$ and

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\| \, d\pi(x, y) = \int_{\mathbb{R}^n} \|x - T(x)\| \, d\mu(x)$$

where $\Pi(\mu, \nu)$ denotes the set of all couplings between μ and ν, and $\|\cdot\|$ is some norm on $\mathbb{R}^n$.

In general T fails to exist.

Beckmann Problem on $\mathbb{R}^n$

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^n)$ be two probability measures on $\mathbb{R}^n$ with finite first moments ($\int_{\mathbb{R}^n} |x| \, d\mu < \infty$, $\int_{\mathbb{R}^n} |x| \, d\nu < \infty$), possessing densities $f_\mu, f_\nu \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with respect to the Lebesgue measure.

Definition (Beckmann Minimal Flow Problem)

The Beckmann minimal flow problem consists of finding:

$$\min_{u \in L^1(\mathbb{R}^n;\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} \|u(x)\| \, dx \;\middle|\; -\operatorname{div}(u) = f_\nu - f_\mu \text{ in } \mathcal{D}'(\mathbb{R}^n) \right\},$$

where the divergence constraint is understood in the sense of distributions:

$$\int_{\mathbb{R}^n} \langle u(x), \nabla \phi(x) \rangle \, dx = \int_{\mathbb{R}^n} \phi(x) \Big(f_\nu(x) - f_\mu(x) \Big) \, dx \quad \forall \phi \in C_c^\infty(\mathbb{R}^n).$$

Equivalence of Formulations on $\mathbb{R}^n$

Assumptions for the Full Space $\mathbb{R}^n$

- **Finite first moments:** $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^n) \implies W_1(\mu, \nu) < \infty$
- **Continuous source:** $\mu \ll \mathcal{L}^n \implies$ mass does not need to split

Theorem (Equivalence of Monge and Beckmann on $\mathbb{R}^n$)

Under these conditions, both problems achieve the **exact same minimum**:

$$\min_{T_\# \mu = \nu} \int_{\mathbb{R}^n} \|T(x) - x\| \, d\mu(x) \;=\; \min_{\substack{u \in L^1 \\ -\operatorname{div} u = \nu - \mu}} \int_{\mathbb{R}^n} \|u(x)\| \, dx \;=\; W_1(\mu, \nu)$$

Monge (Lagrangian view):

- Tracks *who goes where*.
- Pairs source and target points: $x \mapsto T(x)$.

Beckmann (Eulerian view):

- Tracks *aggregate traffic flow*.
- Vector field $u(x)$ gives net flux through space.

Remark: Unlike bounded domains Ω, the whole space $\mathbb{R}^n$ requires no boundary conditions ($u \cdot \vec{n} = 0$).

The Geometric Bridge: $T \longleftrightarrow u$

From Map to Flow ($T \implies u$)

Smear mass along trajectories:

- Send each particle from x to $T(x)$ along a straight segment.
- The flux is defined by testing against ξ:

$$\langle u, \xi \rangle = \int \left(\int_0^1 \xi(\gamma_t) \cdot \dot{\gamma}_t \, dt \right) d\mu(x)$$

  where $\gamma_t = (1-t)x + tT(x)$.
- $\operatorname{div} u = \nu - \mu$ follows directly from the computations.

From Flow to Map ($u \implies T$)

Follow the streamlines:

- Polar factorization freezes the unit directions: $u = \sigma \nabla w^*$.
- Integral curves of ∇w^* are straight **transport rays** that never cross.
- On each 1D ray, conservation of mass yields a **unique monotone map** T_α.
- Gluing T_α gives the global map T.

The Infinite-Dimensional Challenge

The Central Question

Does the equivalence between Monge and Beckmann survive when the ambient space is an infinite-dimensional separable Banach space X?

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Obstacle 1: No Lebesgue Measure

- On an infinite-dimensional space, **no non-trivial translation-invariant Radon measure exists**.
- The classical objective $\int_X |u(x)|\, dx$ is mathematically meaningless.
- **Consequence:** We *must* choose an underlying reference probability measure γ on X, considering fluxes of the form $\int_X |u(x)|\, d\gamma(x)$.

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Obstacle 2: Integration by Parts

To define the divergence constraint $-\operatorname{div}_\gamma(u) = \rho$, we need $\operatorname{div}_\gamma$ as the formal adjoint ∇_γ^* via integration by parts:

$$\int_X \langle u, \nabla\phi \rangle \, d\gamma = \int_X \phi \, \operatorname{div}_\gamma(u) \, d\gamma.$$

Derivative of a Measure

Let $f : X \to \mathbb{R}$ be a smooth function. Define

$$\partial_v f(x) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

Integrating both sides over the measure μ:

$$\int_X \partial_v f(x)\, d\mu(x) = \int_X \left(\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} \right) d\mu(x)$$

Using DCT we swap the limit and the integral:

$$\int_X \partial_v f(x)\, d\mu(x) = \lim_{t \to 0} \frac{1}{t} \left(\int_X f(x + tv)\, d\mu(x) - \int_X f(x)\, d\mu(x) \right)$$

Perform the change of variables (translation) $y = x + tv$ and define the shifted measure $\mu_{tv} := \mu(\cdot - tv)$. We get

$$\int_X \partial_v f(x)\, d\mu(x) = \lim_{t \to 0} \int_X f(y) \left(\frac{d\mu_{tv} - d\mu}{t} \right)(y)$$

A Simple Example of a Singular Shift

Consider a space $\mathbb{R}^\infty$ endowed with a product topology and a measure $\gamma_\infty := \bigotimes_{i=1}^\infty \gamma$ where γ is a standard Gaussian measure $\mathcal{N}(0,1)$. Define a shifted measure $\tilde{\gamma}_\infty := \gamma_\infty(\cdot - \mathbf{1})$ where $\mathbf{1} = (1,1,\dots) \in \mathbb{R}^\infty$.
Note that for any $x = (x_1, x_2, \dots) \sim \gamma_\infty$ the quantity

$$S_N(x) = \frac{1}{N}\sum_{i=1}^N x_n \xrightarrow{N\to\infty} 0 \text{ for } \gamma_\infty\text{-a.e. } x \in \mathbb{R}^\infty$$

by the strong law of large numbers.
At the same time

$$S_N(x+\mathbf{1}) = \frac{1}{N}\sum_{i=1}^N (x_n + 1) = 1 + S_N(x) \xrightarrow{N\to\infty} 1 \text{ for } \gamma_\infty\text{-a.e. } x \in \mathbb{R}^\infty$$

Therefore

$$1 = \gamma_\infty\left(\left\{x \in \mathbb{R}^\infty \mid \lim_{N\to\infty} S_N(x) = 0\right\}\right) \neq \tilde{\gamma}_\infty\left(\left\{x \in \mathbb{R}^\infty \mid \lim_{N\to\infty} S_N(x) = 0\right\}\right) = 0$$

Hence $\tilde{\gamma}_\infty \perp \gamma_\infty$.

Gaussian Measures: Definition

As a reference measure we choose a Gaussian measure on X.

Definition (Gaussian measure)

A Borel probability measure γ on a separable Banach space X is called Gaussian, if for any linear functional $\ell \in X^*$ the image measure $\ell_{\#}\gamma$ is a Gaussian measure on $\mathbb{R}$.

Equivalently, the measure γ is Gaussian, if there exists a vector $m \in X$ (the mean) and a continuous operator $R : X^* \to X$ (the covariance operator) such that for any $g, \ell \in X^*$ one has $\ell(Rg) = g(R\ell)$ and $\ell(R\ell) \geq 0$ and moreover

$$\int_X e^{i\ell(x)}\, d\gamma(x) = \exp\left(i \cdot \ell(m) - \frac{1}{2}\ell(R\ell)\right) \ \forall \ell \in X^*$$

In this case, if $\mathrm{law}(\xi) = \gamma$, we can informally write $\xi \sim \mathcal{N}(m, R)$.

Cameron–Martin Space

Definition (Cameron–Martin Space)

Define a norm $\|\cdot\|_H$

$$\|h\|_H := \sup\{\ell(h) \mid \ell \in X^*, R(\ell)(\ell) \leq 1\}$$

The Cameron–Martin space $H(\gamma)$ is the space of all $h \in X$ such that $\|h\|_H < \infty$.

Theorem (Cameron–Martin formula)

Let γ be a Gaussian measure on X. Then, for any $h \in H$ and $g \in X^*$ such that $R(g) = h$, the measures γ and $\gamma_h := \gamma(\cdot - h)$ are equivalent and the corresponding Radon–Nikodym derivative is given by the expression:

$$\frac{d\gamma_h}{d\gamma} = \exp\left(g(x) - \frac{1}{2}\|h\|_H^2\right)$$

For a further overview we refer to the classical work by V. I. Bogachev.[1]

[1] V.I. Bogachev. *Gaussian Measures*. Mathematical Surveys and Monographs. American Mathematical Society, 2015. ISBN: 9781470418694. URL: https://books.google.ca/books?id=UtufBwAAQBAJ.

Cameron–Martin space as a Canonical Differentiable Structure

Theorem

The directional derivative ∂_v is closable in $L^2(\gamma)$ iff $v \in H$.

Theorem (Intrinsic Invariance of the Cameron–Martin Space)

Let μ be a Gaussian measure. If $X_1 \hookrightarrow X_2$ is a continuous injection between two separable Banach spaces supporting the measure γ, then the Cameron–Martin spaces H_{X_1} and H_{X_2} coincide isometrically as Hilbert spaces.

Moreover, one can define a tangent space structure on X like $TX = X \times H$ (see[2] for instance).

[2] Paul Malliavin. *Stochastic analysis*. Vol. 313. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1997, pp. xii+343. ISBN: 3-540-57024-1. DOI: 10.1007/978-3-642-15074-6. URL: https://doi.org/10.1007/978-3-642-15074-6.

An Abstract Wiener Space

Definition (Abstract Wiener Space)

Let X be an arbitrary separable Banach space, γ be a Gaussian measure on X and H be a real Hilbert space such that $H \hookrightarrow X$ continuously and densely. Then the triplet (X, H, γ) is called an **abstract Wiener space**.[a]

[a] Leonard Gross. "Abstract Wiener spaces". In: *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1.* Univ. California Press, Berkeley, CA, 1967, pp. 31–42.

Abstract Wiener space construction is the only way to obtain a strictly positive Gaussian measure.

Theorem

Let X be an infinite-dimensional separable Banach space and γ be a Gaussian probability measure on it. Then $\gamma(H(\gamma)) = 0$.

The differential structure is completely determined by the measure itself. However, since γ cannot be concentrated on H, we always have to deal with an ambient space X.

The Monge Problem on an Abstract Wiener Space

Definition (Monge problem on an Abstract Wiener Space)

For two probability measures μ, ν with $\mu, \nu \ll \gamma$ find a measurable map $T : X \to X$ such that $T_{\#}\mu = \nu$ and

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} \|x - y\|_H \, d\pi(x, y) = \int_X \|x - T(x)\|_H \, d\mu(x)$$

where $\Pi(\mu, \nu)$ denotes the set of all couplings between μ and ν, and $\| \cdot \|_H$ is a Cameron–Martin norm.

Why? Because, if $T(x) = x + v(x)$ exists, condition $\mu, \nu \ll \gamma$ forces $v(x) \in H$.

Beckmann Minimal Flow on Abstract Wiener Space

Let ∇_H denote the Malliavin (H-)derivative, and $\mathrm{div}_\gamma := \nabla_H^*$ its formal adjoint in $L^2(X, \gamma)$

Definition (Beckmann Minimal Flow on (X, γ))

Let $\mu, \nu \in \mathcal{P}(X)$ such that $(\mu - \nu) \ll \gamma$ with relative density $\rho = \frac{d(\nu-\mu)}{d\gamma} \in L^2(X, \gamma)$. The Beckmann minimal flow problem is:

$$\inf_{u \in L^1(X,\gamma;H)} \left\{ \int_X \|u(x)\|_H \, d\gamma(x) \;\middle|\; \mathrm{div}_\gamma(u) = \rho \right\},$$

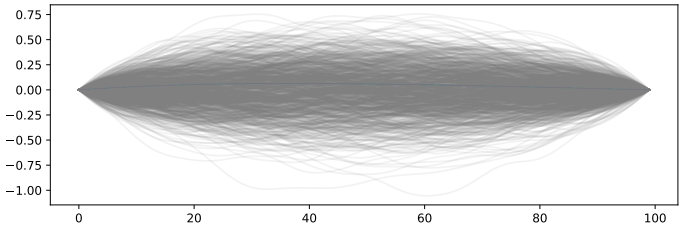
where the constraint $\mathrm{div}_\gamma(u) = \rho$ is understood in the weak sense:

$$\int_X \langle u(x), \nabla_H \phi(x) \rangle_H \, d\gamma(x) = \int_X \phi(x)\rho(x) \, d\gamma(x) \quad \forall \phi \in \mathcal{FC}_b^\infty(X),$$

tested against smooth cylindrical functions $\phi : X \to \mathbb{R}$.

Visual Example of an Infinite-dimensional Distribution

Consider an elastic string with both end fixed. Each possible configuration of a string depends on an infinite amount of parameters. However, some configurations are more likely to happen (mild oscillation), while other are practically impossible. Thus, we have some distribution over the space of string configurations. Moreover, it is an infinite-dimensional distribution.



Analytical solution[3]

Lower Bound ($\leq$ via Duality): By integration by parts and Cauchy–Schwarz, testing against any 1-Lipschitz f ($|Df|_H \leq 1$) yields:

$$\int_X f\, d(\nu_1 - \nu_0) = \int_X \langle Df, u\rangle_H \, d\gamma \leq \int_X |u|_H \, d\gamma \implies W_{1,H}(\nu_0, \nu_1) \leq \inf_{\mathrm{div}_\gamma(u)=\rho} \int_X |u|_H \, d\gamma.$$

Reverse Bound ($\geq$ via OU-Regularization)

1. **Vector Measure on $\mathbb{R}^n$:** It suffices to consider finite-dimensional projections $\rho(x) = \kappa(\hat{e}_1(x), \ldots, \hat{e}_n(x))$ with $\kappa \in L^2_0(\mathbb{R}^n, \gamma_n)$. Finite-dimensional duality yields a vector Radon measure $\pi \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^n)$ such that $\mathrm{div}_{\gamma_n}(\pi) = \kappa\gamma_n$ with total variation $\|\pi\|_v = W_{1,H}(\rho)$.
2. **Mollification:** Convolving π with the Ornstein–Uhlenbeck semigroup $(T_t)_{t>0}$ produces smooth L^2 fields:

$$u_t(x) := e^{-t} \sum_{j=1}^{n} \left(\frac{d(T_t \pi_j)}{d\gamma_n} \right) e_j \in L^2(X, \gamma; H), \quad \mathrm{div}_\gamma(u_t) = T_t \rho.$$

Since $\int_X |u_t|_H \, d\gamma \leq e^{-t}\|\pi\|_v$, taking $t \to 0^+$ proves $\inf \int |u|_H d\gamma \leq W_{1,H}(\rho)$.

[3] G. V. Riabov. "A representation for the Kantorovich-Rubinstein distance defined by the Cameron-Martin norm of a Gaussian measure on a Banach space". In: *Theory Stoch. Process.* 21.2 (2016), pp. 84–90. ISSN: 0321-3900.

Geometrical Solution[4]

Main Theorem

For $\mu, \nu \ll \gamma$ with finite transport cost, there exists a unique optimal γ-essentially invertible Monge map $T : X \to X$.

1. **Hilbert Space Geometry**
   - By $\|\cdot\|_H$-cyclical monotonicity and strict convexity of the Hilbert norm, transport rays cannot cross or branch. Moreover, geodesics are straight lines.
2. **Measure Disintegration:**
   - The disjoint ray interiors form a measurable partition of the transport set.
   - Disintegrate the source measure into 1D conditional probabilities:

$$\mu = \int_Y \mu_\alpha \, d\omega(\alpha).$$

3. **1D Reduction:**
   - On each 1D ray, the non-decreasing transport map T_α is unique.
   - Gluing the 1D maps together yields the global Borel map $T(x) = T_\alpha(x)$.

[4] Fabio Cavalletti. "The Monge problem in Wiener space". In: *Calc. Var. Partial Differential Equations* 45.1-2 (2012), pp. 101–124. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-011-0452-5. URL: https://doi.org/10.1007/s00526-011-0452-5.

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