Kantorovich-Rubinstein Distance on Gaussian Spaces

The Skorohod Readings

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Outline

Introduction

Setup

- We consider a separable Banach space $(X, \|\cdot\|)$ and a centered Gaussian measure μ on the Borel σ-algebra on X.
- Let $(H, |\cdot|_H)$ be the Cameron-Martin space: the separable Hilbert space, densely and continuously embedded in X such that

$$\int_X \exp(i \cdot \ell(x))\, d\mu(x) = \exp\left(-\frac{1}{2}|\ell|_H^2\right), \ell \in X^*$$

- Because of continuous embedding of H into X, a functional $\ell \in X^*$ can be considered as a continuous linear functional on H.
- In the latter expression $|\ell|_H$ denotes the norm of ℓ as an element of H.

Setup

Define the Kantorovich-Rubinstein distance W_1 on the space of Borel probability measures $\mathcal{M}(X)$ over X as

$$W_1(\nu_0, \nu_1) := \inf_{\gamma \in \Gamma(\nu_0, \nu_1)} \int_{X \times X} |x - y|_H \, d\gamma(x, y)$$

where $\Gamma(\nu_0, \nu_1)$ is the set of all joint measures with marginals ν_0 and ν_1.

Theorem

Suppose $\dim H = \infty$*. Then* $|x|_H = +\infty$ *for* μ*-a.e.* $x \in H$*. In particular* $\mu(H) = 0$*.*

Setup

- A function $f : X \to \mathbb{R}$ is called *cylindrical*, if it might be represented as

$$f(x) = \varphi(\ell_1(x), \dots, \ell_d(x)), x \in X$$

for some $\ell_i \in X^*$ and $\varphi \in C_b^\infty\left(\mathbb{R}^d\right)$.
- The set of all cylindrical functions is denoted with $\mathcal{FC}^\infty$.
- Define a *stochastic derivative operator* D for $f \in \mathcal{FC}^\infty$ as

$$(\mathrm{D}f)(x) = \sum_{i=1}^{d} \partial_i \varphi(\ell_1(x), \dots, \ell_d(x)) \cdot \ell_i \in H$$

and extend it to an operator $\mathrm{D} : L^2(X, \mu) \to L^2(X, \mu; H)$
- Define a *stochastic integral operator* I as an adjoint operator of D.

The problem

Theorem (Riabov (2016))

Consider probability measures $\nu_0, \nu_1 \in \mathcal{M}(X)$ *with* $(\nu_1 - \nu_0) \ll \mu$ *and* $\frac{d(\nu_1 - \nu_0)}{d\mu} =: g \in L^2(X, \mu)$. *Then*

$$W_1(\nu_0, \nu_1) = \inf_{\mathrm{I}u=g} \left\{ \int_X |u(x)|_H \, d\mu(x) \right\}$$

Question

Do minimizing function u exist? If it exists, is it unique?

The One-Dimensional Case ($\mathbb{R}$)

The Space and Measure

- Let the target space $X = \mathbb{R}$ equipped with the standard norm.
- Consider a centered Gaussian measure $\mu \sim \mathcal{N}(0, \sigma^2)$ with Lebesgue density:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

- Using the characteristic function of the normal distribution:

$$\int_{\mathbb{R}} \exp(i\ell x)\, d\mu(x) = \exp\left(-\frac{1}{2}\sigma^2\ell^2\right)$$

- Equating this to the canonical form $\exp(-\frac{1}{2}|\ell|_H^2)$, we deduce the norm on the dual space $\mathbb{R}^*$ is $|\ell|_H = \sigma|\ell|$.

The Cameron-Martin Space H

- The Cameron-Martin space H associated with μ is a dense Hilbert subspace of X. Since $X = \mathbb{R}$, as a vector space, $H := \mathbb{R}$.
- However, its geometry is fundamentally altered by the measure μ.
- We find the H-norm by duality:

$$|\ell|_H = \sup_{x \in H, |x|_H \leq 1} |\ell(x)| \implies \sigma|\ell| = \sup_{|x| \leq 1/\sigma} |lx|$$

- This identifies the H-norm and its induced inner product:

Geometry of H in 1D

$$|x|_H = \frac{|x|}{\sigma}, \qquad (x, y)_H = \frac{xy}{\sigma^2}$$

Stochastic Calculus Operators

- **Cylindrical Functions:** For $X = \mathbb{R}$, smooth cylindrical functions $\mathcal{F}C^\infty$ coincide with $C_b^\infty(\mathbb{R})$ (smooth functions with bounded derivatives).
- **Stochastic Derivative (D):** Defined via Riesz representation $\ell(x) = (\ell_x, x)_H$. Applying this yields a scaled derivative:

$$\mathrm{D}f(x) = \sigma^2 f'(x)$$

- **Extended Stochastic Integral (I):** The adjoint $\mathrm{I} = \mathrm{D}^*$ satisfies $\int (u, \mathrm{D}f)_H \, d\mu = \int \mathrm{I}(u) f \, d\mu$.
- Integration by parts against the Gaussian weight $p(x)$ yields:

$$\mathrm{I}(u)(x) = \frac{x}{\sigma^2} u(x) - u'(x)$$

Solving the Divergence Equation

- Consider two probability measures ν_0, ν_1 such that $(\nu_1 - \nu_0) \ll \mu$. We wish to solve $\mathrm{I}u = g$, where $g = \frac{d(\nu_1 - \nu_0)}{d\mu}$.
- Expanding the operator:

$$\frac{x}{\sigma^2} u(x) - u'(x) = g(x)$$

- Multiplying by the integrating factor $p(x)$ collapses the left side into an exact divergence:

$$-\frac{d}{dx}\big(u(x)p(x)\big) = g(x)p(x) = \frac{d(\nu_1 - \nu_0)}{dx}$$

- Integrating yields a **unique** solution in $L^2(\mu)$:

$$u(x) = \frac{F_0(x) - F_1(x)}{p(x)}$$

where F_i are the cumulative distribution functions.

Verifying the solution

- To verify the solution, we evaluate the H-norm of our unique solution u against the measure μ:

$$\int_{\mathbb{R}} |u(x)|_H \, d\mu(x) = \int_{\mathbb{R}} \left| \frac{F_0(x) - F_1(x)}{p(x)} \right|_H p(x) \, dx$$
$$= \int_{\mathbb{R}} |F_0(x) - F_1(x)|_H \, dx$$

- By classical optimal transport theory in 1D, the L^1 distance between CDFs is exactly the 1-Wasserstein distance.

1D Result

$$\int_{\mathbb{R}} |u(x)|_H \, d\mu(x) = W_1(\nu_0, \nu_1)$$

The Multi-Dimensional Case ($\mathbb{R}^d$)

Generalizing to $\mathbb{R}^d$

- Let $X = \mathbb{R}^d$ $(d > 1)$. The centered Gaussian measure μ is defined by a positive-definite covariance matrix A.
- The dual norm, derived from the multivariate characteristic function, becomes $|\ell|_H = \sqrt{\ell^T A \ell}$.
- By duality, the inner product for the Cameron-Martin space $H = \mathbb{R}^d$ is:

$$(x, y)_H = x^T A^{-1} y$$

- The stochastic derivative naturally upgrades to a scaled gradient:

$$\mathrm{D}f(x) = A \cdot \nabla f(x)$$

The Adjoint Operator in $\mathbb{R}^d$

- For a vector field u, we apply multivariate integration by parts against the Gaussian density $p(x)$ to find the adjoint $\mathrm{I} = \mathrm{D}^*$.
- The resulting extended stochastic integral is:

$$\mathrm{I}(u)(x) = (u(x), x)_H - \mathrm{div}(u(x))$$

- We wish to map a source measure to a target measure via the equation:

$$\mathrm{I}u = \frac{d(\nu_1 - \nu_0)}{d\mu} = \frac{p_1(x) - p_0(x)}{p(x)}$$

- Multiplying both sides by $p(x)$ yields a familiar (weighted) continuity equation:

$$\mathrm{div}(u(x)p(x)) = p_0(x) - p_1(x)$$

Beckmann's Problem

- Let $J(x) = u(x)p(x)$ represent the physical mass flux. Our objective function transforms:

$$\int_{\mathbb{R}^d} |u(x)|_H p(x)\, dx = \int_{\mathbb{R}^d} |J(x)|_H\, dx$$

Continuous Max-Flow (Beckmann) Problem

$$\inf \left\{ \int_{\mathbb{R}^d} |J(x)|_H\, dx \;\middle|\; \mathrm{div}(J) = p_0 - p_1 \right\}$$

- The flux J is a vector field that shows how one measure is being transport into other. Given the flux J we can always find a corresponding map flow.

Loss of Uniqueness

- For a solution of $\mathrm{I}u = g$ to be unique, we need $\mathrm{I}u = 0$ to admit only trivial solution $u = 0$. The following example shows that it is not the case in $\mathbb{R}^d$ for $d \geq 2$.

Example

Consider $d = 2$ and and the pure rotation field

$$u(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

- We verify $\mathrm{I}u = (u(x), x)_H - \mathrm{div}(u) = 0 - 0 = 0$.
- Furthermore, $u \in L^2(\mu)$ since $\|u\|_2^2 = \int \|x\|^2 p(x) dx < \infty$.

Conclusion

Any two valid fields differ by a divergence-free component. The infimum eliminates loops, though uniqueness and regularity remains non-trivial.

Tackling existence

- Since unit ball in L^1 is not compact, we need to change the perspective.
- The natural choice is to embedd L^1 into $\mathcal{M}\left(\mathbb{R}^d, \mathbb{R}^d\right) := (C_0(\mathbb{R}^d, \mathbb{R}^d))^*$ (with total variation norm). The embedding is an isometry.
- Given a bounded minimizing sequence $\{J_n\}$, we get a corresponding sequence $\{\mu_n\} \subset \mathcal{M}\left(\mathbb{R}^d, \mathbb{R}^d\right)$. By Banach-Alaoglu Theorem, there exists weakly-$*$ converging subsequence $\{\mu_{n_k}\}$. Denote by μ_* the weak-$*$ limit of $\{\mu_{n_k}\}$.
- By the definition of the weak-$*$ topology, the limit measure natively satisfies the divergence constraint as $\varphi, \nabla\varphi \in C_0(\mathbb{R}^d, \mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \nabla\varphi(x)\, d\mu_* = \lim_{k\to\infty} \int_{\mathbb{R}^d} \nabla\varphi(x) d\mu_{n_k} = \int_{\mathbb{R}^d} \varphi(x)(p_0(x) - p_1(x))\, dx$$

- Semicontinuity of integrals (Fatou's Lemma) guarantees $\|\mu_*\|_{TV}$ attains the minimum.

Summary of the $\mathbb{R}^d$ case

- The topology of L^1 space cannot guarante us neither convergence nor minimality
- By embedding L^1 into $\mathcal{M}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ allows us to use Banach-Alaoglu theorem and lower semicontinuity of integrals.
- Therefore, a minimizer is some **Radon measure**

The Infinite-Dimensional Case (Banach Space)

The Infinite-Dimensional Crisis

- Let X be a separable Banach space. We minimize the total variation of H-valued measures J.
- In general, closed bounded sets are **not locally compact** (Riesz's Lemma).
- The space of functions vanishing at infinity $C_0(X)$ is virtually empty.
- As a consequence, we cannot use the space $(C_0(X))^*$.
- Therefore, the standard functional analysis approach (Banach-Alaoglu) fails. Bounded sequences of measures can "escape to infinity". This hints us at the notion of **tightness**.

Tightness to the Rescue

- Denote by $\mathcal{M}(X, H)$ the space of H-valued Radon measures on the space X.
- Now, instead of being bounded we require the minimizing sequence to be **tight**.
- Since X is a Polish space, by Ulam's Theorem any Borel probability measure is tight. Thus, our marginals ν_0 and ν_1 are supported on compact sets up to an arbitrary $\varepsilon > 0$.
- By Mazur's Lemma, the closed convex hull of compact sets in a Banach space is compact.
- Because optimal transport moves mass along straight lines between the supports of ν_0 and ν_1, we can use compact sets K_0 and K_1 for ν_0, ν_1 to trap each intermediate measure μ_n up to an arbitrary $\varepsilon > 0$.
- Therefore, a minimizing sequence of measures $\{\mu_n\}$ is **uniformly tight**.

Prokhorov's Theorem and The Constraint

- By Prokhorov's Theorem, a uniformly tight sequence of measures has a weakly converging subsequence: $\mu_n \rightharpoonup \mu_*$.
- Notice that by definition, for any $\varphi \in \mathcal{FC}^\infty$, its Cameron-Martin gradient $\nabla_H \varphi$ is also some bounded and continuous function.
- By testing against smooth cylindrical functions $\varphi \in \mathcal{FC}^\infty$ we get:

$$\int_X \varphi(x)\, d(p_0 - p_1)(x) = \lim_{n\to\infty} \int_X \nabla_H \varphi(x)\, d\mu_n(x) = \int_X \nabla_H \varphi(x)\, d\mu_*(x)$$

  Thus the constraint $\operatorname{div}(\mu_*) = \nu_0 - \nu_1$ is satisfied.
- The minimality of μ_* follows from the Fatou's Lemma.

Summary

Summary

- **1D Geometry ($\mathbb{R}$):**
  - The equation $Iu = g$ resolves to a first-order ODE.
  - Integrability constraints force a strictly unique solution.
- **Multi-dimensional Geometry ($\mathbb{R}^d$):**
  - Transforms into the Beckmann continuous transport problem.
  - Requires weak-$*$ topology to guarantee existence of a measure minimizer.
- **Infinite Dimensions (Banach Space):**
  - We no longer can use weak-* topology.
  - Lack of local compactness requires Prokhorov's Theorem and Tightness.
  - However, minimizing measure still exists.

Open Questions

Question 1

Is minimizing measure μ_* absolutely continuous? If no, under which conditions?

Question 2

What is support of μ_*?

Question 3

Under which conditions on measures ν_0, ν_1 the minimizing measure is unique?

Thank You for Your Attention